Assignment 10.

Taylor Series. Uniqueness Theorem.

This assignment is due Wednesday, April 6. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

The function $\ln(1+z)$ is as introduced in HW9.

(1) Prove the following theorem.
Theorem. *Suppose the series*

$$f(z) = \sum_{k=1}^{\infty} f_k(z)$$

is uniformly convergent on every compact subset of the disc $K : |z - z_0| < R$, *and suppose that every function* $f_k(z)$ *is analytic on* K. *Then* $f(z)$ *has the Taylor series expansion* $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, *where*

$$a_n = \sum_{k=1}^{\infty} \frac{f_k^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, \ldots).$$

(*Hint:* Use Weierstrass theorem on uniformly convergent series of analytic functions and Taylor expansion theorem.)

COMMENT. Since $\frac{f_k^{(n)}(z_0)}{n!}$ is n-th Taylor coefficient of f_k, this theorem explains the following fact: if Φ and f are analytic, then Taylor series for $\Phi(f)$ can be obtained by formal substitution of Taylor series of f into the Taylor series of Φ. You can take this fact for granted in Problems 2, 3.

(2) Find Taylor series at $z = 0$ and its radius of convergence of the following functions.
(a) $\sin^2 z$; (b) $\cosh z^5$; (c) $\frac{1}{az+b}$ $(a, b \in \mathbb{C}, b \neq 0)$; (d) $\frac{1}{z^2-5z+6}$;
(e) $\int_0^z e^{\zeta^2} d\zeta$; (f) $\int_0^z \frac{\sin\zeta}{\zeta} d\zeta$; (g) $\ln(1 + z^4)$.

(3) Find the terms up to degree 5 in the Taylor expansion at 0 of the following functions. (Either by computing derivatives, or by formal substitution, as explained above[1])
(a) $e^{z\sin z}$; (b) $(1 + z)^z = e^{z\ln(1+z)}$; (c) $\cos(z^3 + 1)$; (d) e^{e^z}.

(4) (a) Prove that the coefficients c_n of the expansion

$$\frac{1}{1 - z - z^2} = \sum_{n=0}^{\infty} c_n z^n$$

satisfy the recurrence relation

$$c_0 = c_1 = 1, \ c_n = c_{n-1} + c_{n-2},$$

by multiplying $(1 - z - z^2)\sum_{n=0}^{\infty} c_n z^n$.

(b) Expand $\frac{1}{1-z-z^2}$ in a Taylor series at 0 by decomposing $\frac{1}{1-z-z^2}$ into partial fractions.

COMMENT. We just got an explicit formula for Fibonacci numbers.

COMMENT. One can observe that, similarly to (a), a linear recurrent relation takes place for coefficients for Taylor series of arbitrary rational function.

— see next page —

[1] I think substitution is faster (and more interesting).

(5) Does there exist a function that is analytic on a neighborhood of $z = 0$ and takes the following values at $z = 1/n$ $(n = 1, 2, \ldots)$:
   (a) $0, 1, 0, 1, 0, 1, 0, 1, \ldots$;
   (b) $0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, \ldots, 0, \frac{1}{2k}, \ldots$;
   (c) $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \ldots, \frac{1}{2k}, \frac{1}{2k}, \ldots$;
   (d) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots, \frac{n}{n+1}, \ldots$?
   (*Hint:* Use the interior uniqueness theorem.)

(6) Does there exist a function that is analytic on a neighborhood of $z = 0$ and satisfies the following condition for every positive n:
   (a) $f(1/n) = f(-1/n) = 1/n^2$ (*Hint:* Yes);
   (b) $f(1/n) = f(-1/n) = 1/n^3$ (*Hint:* No)?